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LARGE HYPERBOLIC FUNCTIONS COMPUTED BY FISSION

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STRUCTURAL DIVISION

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PAPERS

LARGE HYPERBOLIC FUNCTIONS COMPUTED BY FISSION

By F. T. Llewellyn,[1] M. ASCE

Synopsis

A rapid "fission" method is presented for computing the hyperbolic sines and cosines of fractional radians larger than are listed in current reference tables. Such functions are often needed by structural engineers when solving cubic and quartic equations (notably in problems requiring the determination of maxima and minima) by a direct procedure such as those of Messrs. Cardan,[2] Ferrari,[3] or Euler.[4] The primary objective is to simplify the work of practical designers and computers.

Within the range of widely spaced arguments larger than about six radians, the precision of the fission method is much greater than that of interpolation by proportional parts, and adequate precision is attained much more readily than by academic methods. The scope of several current sets of reference tables is surveyed and analyzed. Examples illustrate the preferred application of the slide rule method.

Introduction

Important phases of engineering design can often be expedited by the use of "hypes" (hyperbolic sines and cosines) larger than those listed in current tables of reference. The nonlisted functions at issue are the hypes of fractional arguments greater than about six radians. Their lack may explain in part why the advantages of a direct procedure, notably in solving cubic and quartic equations, have not been accorded fuller recognition by civil engineers.

Note.—Written comments are invited for publication; the last discussion should be submitted by June 1, 1951.

[1] Baton Rouge, La.

[2] "Smithsonian Mathematical Tables," by G. F. Becker and C. E. Van Orstrand, Smithsonian Institution, Washington, D. C., 1909, p. 26.

[3] "Transmission Circuits for Telephonic Communication, Methods of Analysis and Design," by K. S. Johnson, D. Van Nostrand Co., Inc., New York, N. Y., 1925, p. 249.

[4] "Higher Algebra," by G. E. Fisher and I. J. Schwatt, Macmillan Co., New York, N. Y., 1924, p. 614.

This study seeks to simplify direct procedure in such solutions by presenting an efficient method for computing the functions of any large radian in probable demand, and certainly any that may be expected in structural and allied problems. In so doing certain limitations that mark the standard process of interpolation are analyzed with what is thought to be a new approach. As far as the writer can find, the predictability of degrees of precision, as developed in this paper, has not hitherto been evaluated. The study is intended primarily for designers who cannot afford to interrupt the solution of a specific problem to "brush up" on their higher mathematics. Its objective is adequate precision combined with minimum labor, rather than unnecessarily high precision. Emphasis is therefore laid on the adaptation of natural functions and a slide rule, thereby obviating the use of logarithms.

Background

About 1926 the late O. E. Hovey, Hon. M. ASCE, while preparing the manuscript for his classic treatise on the design of bridge machinery, found that certain fourth-power examples,[5] if solved by a direct and complete method (such as those of Ferrari or Euler), would entail the meticulous computation, by log logs, of the hypes of large unlisted radians. (The "radian," of course, is a unit of circular measure equal to $180/\pi$, or 57.29578°; but it is also used as a general term for the argument of a function. For example, "large" radians are radian arguments whose sinh and cosh hypes are substantially equal. Radians greater than 6.0, or the hypes of such radians, are considered "large.") Such a procedure was then regarded by the average designer as too formidable for everyday use. Mr. Hovey told the writer that this was one of several reasons why he had substituted an indirect and partial "trial-and-error" method of solution. The episode aroused in the writer a continuing interest in the direct solution of such problems and, hence, in hypes. (The term, "hype," denotes any hyperbolic function, although in this paper only sinh hypes and cosh hypes are treated.)

For instance, shortly thereafter he completed the direct solution[6] of Mr. Hovey's examples with the aid of large hypes; but the fission method of computing them was not then realized. (The fission method, described in this paper, is a process of computing any large unlisted hype by splitting its radian into two listed parts—or vice versa—and then combining their listed *e*-functions by multiplication or division. The convenient and accurate term "fission" was not borrowed from recent accounts of atomic reaction. Its use had long been anticipated elsewhere, as in the fields of botany and geology.) In 1935 he privately circulated 100 copies of 8 cubics and quartics, similarly solved; and in the succeeding fifteen years (in industry and research) he solved more than fifty such problems wherein the radians ranged from 0.0017 to 16.473, with eleven radians being classed as "large." He had previously observed that large unlisted hypes arise in the Cardan solution of many third-power equations, and elsewhere.

[5] "Movable Bridges," by O. E. Hovey, John Wiley & Sons, Inc., New York, N. Y., 1926–1927, Vol. II, Chapter 2, Examples B2 and B3.

[6] "Solving Cubic and Quartic Equations," by F. T. Llewellyn, *Engineering News-Record*, August 14, 1930, pp. 254–256.

Survey of Tables

A logical study of large hypes cannot disregard the behavior of small ones, and, hence, the entire scope of a number of available reference tables was surveyed and analyzed. Table 1 is a semi-idealized frame in which are summarized the relevant data from six sets of "current" tables containing natural functions. (In the sense used in this paper, "current" tables are the mathematical tables commonly accessible to civil engineers.) Some of their groupings

TABLE 1.—Survey of Current Hyperbolic Tables

Group	Scope of Tables				Interpolation[a] of Hype y	
	Range of Radians		Listed intervals,	Nature of functions	Maximum error,[b] R_E (%)	Critical[a] factor p_c
	x_{min}	x_{max}	v_{min}			
(1)	(2)	(3)	(4)	(5)	(6)	(7)
(a) Small Radians; sinh x and cosh x Are Substantially Different[d]						
1	0.0	0.1	0.0001	e^x and hypes	12.5 (10^{-3})	0.500
2	0.1	3.0	0.001	e^x and hypes	12.6 (10^{-6})	0.499[c]
3	3.0	6.0	0.01	e^x and hypes	12.7 (10^{-4})	0.497
(b) Large Radians; sinh x and cosh x Are Substantially Equal[e]						
4	6.0	10.0	0.1	e^x and hypes	12.9 (10^{-3})	0.492
5	10.0	100	1.0	e^x only	13.1 (10^{0})	0.418
(c) Hypothetical Radians; Extensions by Eqs. 5 and 6						
6	none	none	5 (assumed)	neither e^x nor hypes	10 (10^{2})	0.193
7	none	none	10 (assumed)	neither e^x nor hypes	8 (10^{4})	0.100

[a] The values in Cols. 6 and 7 are approximate, being determined by the interpolation of hype y by proportional parts (PP). [b] In groups 2 to 5, Col. 6, each R_E-ratio is approximately one hundred times that of the next smaller group. [c] The critical p-curve (Fig. 1) appears to approach $p = 0.5$ as x and v approach zero. As v approaches infinity, the curve is definitely asymptotic to $p = 0$. [d] Both logarithms and naturals in each set. [e] Naturals in two sets and logarithms in one set.

overlap, so that the combined scope of a plurality of tables is depicted. Complete references, and a discussion of the six sets of tables, are given in Appendix I; and the mathematical derivations are presented in Appendix II. Referring to Table 1, let:

e = base of natural (Naperian) logarithms = 2.71828 +;
M = magnitude of hype y:

M_P = the value of M found by proportional parts (PP), or

$$M_P = \text{hype } x + p\,[\text{hype } (x + v) - \text{hype } x] \quad \ldots\ldots\ldots\ldots (1a)$$

M_T = the true value of M; that is,

$$M_T = \text{hype } y = \text{hype } (x + p\,v) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1b)$$

p = a nondimensional factor of v in the part $p\,v$ to be interpolated:

p_c = the critical value per group of p corresponding to R_{max}; thus, R equals a general ratio of M_P/M_T, as expressed by the basic formula:

$$\frac{\text{Eq. } 1a}{\text{Eq. } 1b} = \frac{\text{hype } x + p\,[\text{hype } (x + v) - \text{hype } x]}{\text{hype } (x + p\,v)} \quad \ldots\ldots\ldots (1c)$$

R_E = the maximum ratio of the proportional parts (PP) error ($R_{max} - 1$); multiplied by 100, it is expressed as a percentage;

v = the interval between any two listings of x, in radians;

x = any listed radian, in circular measure; and

y = any unlisted radian ($x + p\,v$) whose hypes are sought.

Col. 6, Table 1, shows the approximate percentages of maximum PP-error by groups. In certain respects group 4 in Table 1(b) may be classed as "small," but the upper limit of small radians was held at $x = 6.0$ because the more popular tables detail no hypes beyond that limit. Groups 6 and 7 (Table 1(c)) do not appear as such in current tables. They are included in Table 1 because their trends were helpful in establishing the critical p-curve plotted in Fig. 1 and discussed in Appendix II.

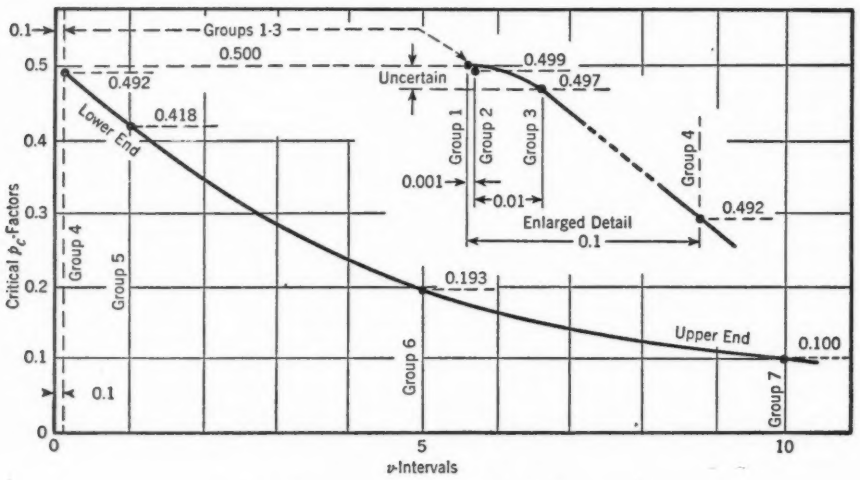

FIG. 1.—GRAPH OF CRITICAL p_c-CURVE (INVERTED)

In two sets of tables (discussed subsequently under the heading, "Fission Method" and in Appendix I) special cognizance was taken of a series of natural e^x-entries against radians ranging up to 100, which is much greater than any argument the writer has met in practice. In group 5 their listed interval ($v = 1.0$) would be too large for PP-interpolation, but it was correctly surmised that even these widely spaced entries might suffice to implement some method of computation yet to be developed. However, before considering any new

methods, existing procedures were subjected to two lines of scrutiny, suggested by the questions:

1. Is there any practical substitute for hyperbolic tables? and
2. To what extent do current tables provide for nonlisted functions?

The first question does not warrant long discussion. A convergent series of the $y^{f(n)}$-type might indeed be carried (generally by the often puzzling and always irksome use of log logs) to any desired number of terms $(1 + r)$. (This type of convergent series was used to find the maximum PP-error in group 1.) Again, the magnitude of any hype could be found by a single evaluation of the $e^{f(y)}$-type, which sums up another series and also entails log logs; but academic procedures of this kind would tend to distract the attention of busy designers from the design in hand. They are doubtless useful to mathematicians when compiling permanent tables, and in research work, but they are much too prolix to replace published tables of reference for everyday use.

The second question could not be dismissed so summarily. To be sure, it suggested the time-honored process of interpolation by proportional parts, which is fairly simple to apply, but its reliability is not uniform. Its R_E-ratios increase sharply by groups as the v-interval increases, whereas (within each group) its R-ratios rise and fall in waves whose crest changes, both in position and size, as the p-factor increases. One rigorous characteristic is common to all groups; when $p = 0.5$, $R = \cosh p\,v$ (see Appendix II). The method followed in approximating the critical value of the other ratios calls for some particulars and a frank admission.

In the general case, the writer had been unable to manipulate rigorously the basic R-formula, Eq. 1*c*, with its three variables (see Appendix II) so as to locate the critical value of p_c corresponding to $R_{\max}$ and hence for R_E. Therefore, the writer started with the special case of large radians and intervals, and in that range he found that concise formulas, having ample precision, were derivable, and that they would fix the upper part of the critical p-curve (Fig. 1). In their development the following identity was assumed:

$$2 \sinh x \doteq e^x \doteq 2 \cosh x \qquad (2)$$

At $x = 6.0$ the error in this assumption is less than 0.001%, and it diminishes as x increases. By treating the assumption as exact it is readily seen that, when large hypes are interpolated by PP, $f(x)$ vanishes, and the ensuing relations reduce quite simply from Eq. 1*c* to terms of e, v, and p, with only two variables. For example (see Appendix II), the general formula for any large group becomes

$$R = \frac{1 + p\,(e^v - 1)}{e^{pv}} \qquad (3)$$

A characteristic common to all groups is that, when $p = 0.5$, Eq. 3 becomes

$$R = \cosh\,(p\,v) \qquad (4)$$

For the critical factor—

$$p_c = \frac{1}{v} - \frac{1}{e^v - 1} \qquad (5)$$

—the maximum value of Eq. 3 is

$$R_{\max} = \frac{e^v - 1}{v\, e^{pv}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and, in group 5, with—

$$p_c = \frac{e - 2}{e - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

—the maximum value of Eq. 3 becomes

$$R_{\max} = \frac{e - 1}{e^p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Eqs. 3 and 6 apply very closely for both sinh hypes and cosh hypes in group 5 (the principal range at issue); they apply quite well in group 4; and, to some extent, even in group 3. The R-ratios as thus forecast can be checked most easily near the boundaries between these three groups, and by the check for nonconsecutive entries suggested subsequently. It will be noted from Col. 6, Table 1, that, in group 5, the possible error ($R_E = 13.1\%$) is much greater than safe practices would tolerate.

In the case of very small radians, however, $f(x)$, instead of vanishing, becomes so dominant that Eqs. 3 and 6 no longer apply, except at $p = 0.5$. Since the basic R-equation (Eq. 1c) seemed here to be unworkable, recourse was taken to a semi-empirical procedure. By Eq. 6, and by trial, the critical p_c-curve (Fig. 1) had straightened out for groups 4 and 3. It was also noted that p_c seemed to approach 0.5 as x and v approach zero, and this value was assumed for entire group 1. The simplest reverse curve was then "faired" by eye, tangential to the curve at group 3, and to the horizontal at group 1. This curve gave $p_c = 0.499 +$ as the nearest three-digit value (less than 0.5) for group 2. The respective R_E-ratios were finally adjusted to correspond, and are so entered in Table 1.

These ratios show tenable trends at the lower end of the curve but they are too small for close checking, even by seven-place log tables, and their true values are uncertain. The topic is intriguing but purely academic because the amount of the PP-error here becomes microscopic. To summarize, the group values in Fig. 1 were plotted as follows:

Groups	Plotted by
3 to 7	Eq. 6
1	analogy
2	eye

Groups 3 to 7 were checked by trial, group 2 was checked partly by trial, and group 1 was too small to be checked.

Of more practical significance as applied to Eqs. 3 and 6 is the fact that v is not restricted to the intervals that characterize groups 3 to 7 in Table 1; nor is p restricted to factors less than unity. Thus, these equations satisfy any interpolation, or indeed any upward extrapolation, within or outside any pair of large or semilarge radians. The validity of Eq. 3 and of Eq. 6 can thus be checked against nonconsecutive entries.

When the main problem is reversed, as in computing radians from hypes, R is subject to considerable reduction because those terms that might have adverse effects can then be thrown into a relatively small addendum, $f(pv)$. This truism is applicable to any approximate method. Subsequently, under the heading, "Fission Method," Example 2 illustrates the closer precision attainable by both PP and fission in this type of problem.

In either type the survey, as summarized in Table 1, definitely indicated that the PP-process of interpolation, although entirely adequate for small radians, might lead to excessive error for the large ones at issue. In the former ranges the exact amount of error was theoretically elusive but is practically negligible, whereas in the latter it was readily ascertainable and is important. It was concluded that either more comprehensive tables would be necessary, or a better method must be sought for handling large radians and hypes.

Fission Method

Further study of this dilemma led to the development of a procedure that is based on well-known theory, but whose practical application may be new to many engineers. It asks for no elaboration of existing tables. By this process the unlisted hypes of any large radian (up to $y = 100$) can be interpolated very rapidly (but not by PP) from current natural tables that list both large and small e^x-values and are otherwise in line with Table 1. With the aid of an ordinary slide rule, such tables permit results that are correct to at least four significant digits when hypes are to be computed from radians, or to three places, and sometimes four places, when radians are to be found from hypes. The process entails much less labor than computation by $e^{f(w)}$, and somewhat less than by the (here) unsuitable PP-method.

The writer has named the suggested procedure a "fission" method, because it splits the nonlisted radian (y) into two listed parts (a and b). When $y = a + b$, current textbooks show that

$$\cosh y = \cosh a \cosh b + \sinh a \sinh b \quad \ldots\ldots\ldots\ldots\ldots (9)$$

Also, when $a > 6.0$,

$$\sinh a \doteq \cosh a \doteq \frac{e^a}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Therefore, when a (and hence y) is large, Eq. 9 simplifies to

$$\text{hype } y \doteq e^b \text{ hype } a \doteq \frac{e^b e^a}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which hype denotes either sinh or cosh. Eq. 11 works also inversely, when it takes the form:

$$e^b \doteq \frac{e^y}{e^a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Eq. 9 is a general expression for splitting any cosh hype.

If logs were to be used, a and b could have any listed values. However, when natural tables and the slide rule are used, b should be kept small enough

to fall into such tables as give the most decimals. Under the following rule, b is thus thrown into group 1 or 2 (which give radians to four decimals or three decimals), whereas a will remain in group 4 or group 5, thus: a (listed) $= x$ = integral part of y; and b (listed) $= p\,v$ = fractional part of y.

The handiest tables list values of e^b alongside hypes of b. In case such tables are inaccessible the value sought can be found, but less readily, from the following identity in which, for small radians, sinh b and cosh b are markedly unequal:

$$e^b = \sinh b + \cosh b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Two rather extreme examples may suffice to illustrate the precision and brevity of the fission method. Its trivial aberrations are caused almost entirely by slide rule "roundings." No problem needs more than two tabular references and one slide rule setting.

Example 1 (Direct); Compute sinh 28.49 (Unlisted).—By the rule for a and b, $a = 28$ (listed) and $b = 0.49$ (listed). Eq. 11, then, yields (by slide rule): $\sinh y = \frac{1}{2}\, e^{0.49}\, e^{28} = 1.632 \times 723 \times 10^9 = 11{,}800 \times 10^8$.

Example 2 (Reverse); Compute y (Unlisted), from cosh y = 713 × 10⁴.—In this case $a = 16$, which is the integer-radian x tabulated next preceding y, in which (compare Eq. 2) $e^v = 2 \cosh y = 1{,}426 \times 10^4$. Consequently, e^a (listed) $= 8{,}886 \times 10^3$. By Eq. 12 (using slide rule), $e^b = e^v/e^a = 14{,}260/8{,}886 = 1.605$. Then b (listed) $= 0.473$ and $y = 16.473$.

TABLE 2.—Relative Precision of Methods

Method	Example 1		Example 2	
	sinh y	Error (%)	Radian y	Error (%)
Logarithmic (seven place)	11,804 (10^8)	Norm	16.473	Norm
Fission (10-in. slide rule)	11,800 (10^8)	0.034	16.473	0.00
Proportional parts (10-in. slide rule)	13,320 (10^8)	12.8	16.352	0.73

Table 2 compares three of the methods herein discussed as applied to the two examples. The method marked "logarithmic" furnishes norms of precision against which the other two methods are rated. These norms were calculated by the meticulous $e^{f(y)}$-procedure and checked by seven-place log logs with Eqs. 11 and 12. The negligibility of fissional errors is apparent immediately. The importance of adequate precision is properly emphasized because even a small error in hype magnitudes may have a considerable effect on the final design, especially in those rather frequent cases in which the critical difference between two large quantities proves to be relatively small.

Summary

The large functions to whose computation the fission method is directed are the hypes of fractional radians lying between 6 and 100. Within this range the entries listed in current tables can be combined by slide rule at a single step to find any intermediate unlisted hypes with a precision substantially that of their listed sources. In its labor-saving slide rule work the method

calls for no special skill to meet the ordinary needs of civil engineers. Those who prefer to use log logs can apply the fission concept as a quick independent check.

The fission method is not applicable to cases in which radians are much less than 6; but in those cases the old process of interpolation by proportional parts is fortunately available, and is adequate.

Thus, the unlisted, as well as the listed, hypes of all radians in normal demand become readily determinable. In the field of cubic and quartic equations having real coefficients, the writer has shown, elsewhere,[6] the universal feasibility of a direct method of solution by hypes (or in one type by trigs), and has emphasized its general advantages. (By printer's error, a radical $\sqrt{3A + y_1}$ appeared as $\sqrt{3A - y_1}$ in that work.[7]) To these advantages the special relief afforded by fission may now be added. The writer has not investigated the extent to which the fission method may be desirable in other fields, but its practical scope appears to be quite broad.

APPENDIX I. DESCRIPTION OF CURRENT SETS OF NATURAL HYPERBOLIC TABLES

The six sets of tables containing natural functions that were examined in detail are listed chronologically in Table 3. Each one appeared to be excellent in its respective fields, but not equally well suited to the objectives of the fission method. In sets 3 to 6 (Tables 3(*c*), 3(*d*), 3(*e*), and 3(*f*)) some entries were either too widely spaced or too limited in range to meet all cases readily. This was notably true in the otherwise meticulous British set, set 3 (fifteen decimal places), which was found only in very large scientific libraries.

However, it was the writer's experience that sets 1 and 2 (Tables 3(*a*) and 3(*b*)) were fully suited to the fission method. Their hype entry and $f(e)$-entry comply essentially with the ideals of Table 1, and they were the only sets found containing group 5 (Table 1), the principal range at issue.

For general purposes the Smithsonian tables are the more comprehensive. They are best suited to logarithmic computation. For fission by slide rule they may entail reference to three dissociated pages of the book.

The Hayashi tables[8] contain only natural functions and are thus restricted to slide rule computation. (Note: There is an error in the Hayashi tables[8] where the factor for e^{50} should read "10^{13}" and not "10^{14}.") Most of the information needed for a given problem is assembled on a single pair of facing pages, only one additional crossreference being required. This handy little book, although published abroad, is classed as "current" because copies were on general sale in New York, N. Y., prior to World War II, and are doubtless still available in the United States.

The only known previous literature on the practical objectives of the present study consists of brief statements in the prefaces of some of the tables of reference, none of which, however, suggest fission.

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[7] "Solving Cubic and Quartic Equations," by F. T. Llewellyn, *Engineering News-Record*, August 14, 1930, p. 254, step 10.

[8] "Funfstellige Tafeln der Kreis, und Hyperbelfunktionen," by K. Hayashi, Walter de Gruyter and Co., Berlin, 1921, p. 175.

TABLE 3.—Pertinent Details of Six Current Sets of Natural Hyperbolic Tables[a]

Pages (inclusive)	Radians: Range		Minimum interval	Functions		Decimal places
	Minimum	Maximum		Hype (=h) e^x (=e)	Natural (=N) or logs (=L)	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
(a) Set 1.—"Smithsonian Mathematical Tables," (Hyperbolic Functions) by G. F. Becker and C. E. Van Orstrand, Smithsonian Institution, Washington, D. C., 1909.						
2 to 21	0	0.1	0.0001	h	L	Five
22 to 79	0.1	3	0.001	h	L	Five
80 to 85	3	6	0.01	h	L	Five
88 to 171	[b]	[b]	[b]	h	N	Five
226 to 255	0	3	0.001	e	L, N	Seven and six
256 to 258	3	6	0.01	e	L, N	Seven and six
259 to 260	1	100	1	e	L, N	Integers
261 to 262	0	0.5	0.001	e	L	Auxiliary
(b) Set 2.—"Funfstellige Tafeln der Kreis, und Hyperbelfunktionen," by K. Hayashi, Walter de Gruyter and Co., Berlin, 1928.						
2 to 41	0	0.1	0.0001	h, e	N	Five and six
42 to 157	0.1	3	0.001	h, e	N	Five and six
158 to 171	3	6.3	0.01	h, e	N	Five and six
171 to 173	6.3	10	0.1	h, e	N	Five and six
174	0	1	varies	e	N	Twelve, ten, and eight
175	1	10	1	e	N	Eight
175 to 176	11	100	1	e	N	Integers
(c) Set 3.—"Mathematical Tables," British Assn. for the Advancement of Science, University Press, Cambridge, England, 1931, Vol. 1.						
30	0	10	0.1	h[c]	N[c]	Fifteen[c]
(d) Set 4.—"Tables of Circular and Hyperbolic Sines and Cosines," by a Committee of the Works Progress Administration, National Bureau of Standards, Washington, D. C., 1937.						
1 to 400	0	2	0.0001	h	N	Nine
402 to 403	0	10	0.1	h	N	Nine
(e) Set 5.—"Mathematical Tables of Elementary and Some Higher Mathematical Functions Including Trigonometric Functions of Decimals of Degrees and Logarithms," by H. B. Dwight, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.						
128 to 147	0	5	0.001	e	N	Nine
148 to 165	0	3	0.001	h	N	Nine
(f) Set 6.—"Six-Place Tables," by Edward S. Allen, McGraw-Hill Book Co., Inc., New York, N. Y., 1947.						
172 to 191	0	1	0.001	h, e	N	Six
192 to 201	1	2	0.002	h, e	N	Six
202 to 205	2	3	0.005	h, e	N	Six
206 to 208	3	6	0.01	h, e	N	Six

[a] General reference: "An Index of Mathematical Tables," by A. Fletcher, J. C. P. Miller, and L. Rosenhead, McGraw-Hill Cook Co., Inc., New York, N. Y., 1946 (an analytical summary including hyperbolic tables). [b] Same as the foregoing three groups. [c] Other functions not pertinent.

APPENDIX II. MATHEMATICS

General Relations at $p = 0.5$.—By trigonometry,

$$\sinh (x + v) = \sinh x \cosh v + \cosh x \sinh v \quad \ldots\ldots (14)$$

$$\cosh (x + v) = \cosh x \cosh v + \sinh x \sinh v \quad \ldots\ldots (15)$$

$$\tfrac{1}{2} (1 + \cosh v) = \cosh^2 \frac{v}{2} \quad \ldots\ldots (16)$$

and

$$\sinh v = 2 \sinh \frac{v}{2} \cosh \frac{v}{2} \quad \ldots\ldots (17)$$

The basic R-formula Eq. 1*c* is then written—

$$R \text{ (sinh)} = \frac{M_P}{M_T} = \frac{\sinh x + p\ [\sinh (x + v) - \sinh x\ ]}{\sinh (x + p\, v)} \quad \ldots\ldots (18)$$

At $p = 0.5$—

$$\left.\begin{aligned} M_P &= \frac{\sinh x}{2} + \frac{\sinh x \cosh v}{2} + \frac{\cosh x \sinh v}{2} \\ M_P &= \sinh x \frac{1 + \cosh v}{2} + \frac{\cosh x}{2} 2 \sinh \frac{v}{2} \cosh \frac{v}{2} \\ M_P &= \sinh x \cosh^2 \frac{v}{2} + \cosh x \sinh \frac{v}{2} \cosh \frac{v}{2} \\ M_P &= \cosh \frac{v}{2} \left[\sinh x \cosh \frac{v}{2} + \cosh x \sinh \frac{v}{2} \right] \\ M_P &= \cosh \frac{v}{2} \sinh \left(x + \frac{v}{2} \right) = \cosh \frac{v}{2} M_T \end{aligned}\right\} \quad \ldots\ldots (19)$$

Then R (sinh) $= \cosh \frac{v}{2} = \cosh p\, v$ (see Eq. 4). Similarly,

$$\left.\begin{aligned} M_P \text{ (cosh)} &= \cosh \frac{v}{2} \left(\cosh x \cosh \frac{v}{2} + \sinh x \sinh \frac{v}{2} \right) \\ M_P \text{ (cosh)} &= \cosh \frac{v}{2} \cosh \left(x + \frac{v}{2} \right) = \cosh \frac{v}{2} M_T \end{aligned}\right\} \quad \ldots\ldots (20)$$

Then R (cosh) $= \cosh \frac{v}{2} = \cosh p\, v$ (see Eq. 4). Thus, in all groups, and for both sinh hypes and cosh hypes at $p = 0.5$, $R = \cosh p\, v$. At $p = 0.5$, however, R is not (Table 1) generally a maximum. It is almost a maximum in groups 1 to 4; but in group 6 it drops to 56%, and in group 7 to only 9.2% of the maximum.

Eq. 3 (x Large).—When x is large Eq. 1*c*, as simplified by Eq. 2, becomes

$$R \doteq \frac{e^x + p\ (e^{x+v} - e^x)}{e^{x+pv}} \quad \ldots\ldots (21)$$

Clearing e^x, $R = \dfrac{1 + p\,(e^v - 1)}{e^{pv}}$ (see Eq. 3).

Eq. 6 (x Large).—In Eq. 3, e is constant, and v is constant per group, and thus R and p are the only variables. By calculus, R is a maximum (or minimum) when its first derivative (with respect to p) is zero. Here dR gives

$$e^{pv}\,d\,[1 + p\,(e^v - 1)] = [1 + p\,(e^v - 1)]\,d\,e^{pv} \quad \ldots\ldots\ldots (22)$$

Hence, $e^{pv}\,(e^v - 1) = [1 + p\,(e^v - 1)]\,v\,e^{pv}$. Dividing by the left-hand member,

$$1 = \frac{v}{e^v - 1} + p\,v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

and, dividing by v, the critical value of $p_c = \dfrac{1}{v} - \dfrac{1}{e^v - 1}$ (see Eq. 5), in which p_c is positive in interpolation and in upward extrapolation

Since the second derivative of R here becomes negative, this point of inflection denotes a maximum. (By inspection, minimum R (= 1) occurs at two points ($p = 0$ and 1), but they are not points of inflection.) Then, by Eqs. 3 and 5,

$$R_{\max}\ (x \text{ large}) = \frac{1 + \dfrac{e^v - 1}{v} - 1}{e^{pv}} = \frac{e^v - 1}{v\,e^{pv}} \quad \ldots\ldots\ldots\ldots (6)$$

Lower End of p_c-Curve.—The statement in the text that Eq. 1c seemed to be unworkable for the general case does not mean, of course, that it could not be differentiated, but rather that the resulting expression became too complex to justify its rigorous solution. It definitely indicated, however, that for small radians p_c would vary with each entry, and not per group.

Since trial had shown that R_E was here quite negligible, it seemed sufficient to indicate reasonable trends at the lower end of the curve as in Fig. 1. An alternative hypothesis might be that $p_c \doteq 1.0$ instead of 0.5 for group 1, Table 1, which would change the lower end of the curve, but the PP-error for small radians would remain negligible.

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